J Glob Optim (2010) 46:331–345
DOI 10.1007/s10898-009-9428-9

Ishikawa iterative algorithms for a generalized equilibrium problem and fixed point problems of a pseudo-contraction mapping

Jian-Wen Peng · Jen-Chih Yao

Received: 8 November 2008 / Accepted: 25 April 2009 / Published online: 12 May 2009

Abstract In this paper, we propose two Ishikawa iterative algorithms for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a Lipschitz continuous pseudo-contraction mapping. We obtain both strong convergence theorems and weak convergence theorems in a Hilbert space.

Keywords Generalized equilibrium problem · Hybrid method · Pseudo-contraction mapping · Strong convergence · Weak convergence · Ishikawa iterative algorithm

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$ and let C be a nonempty closed convex subset of H. Let $B : C \to H$ be a nonlinear mapping and let F be a bifunction from $C \times C$ to R, where R is the set of real numbers. Moudafi [1], Moudafi and Thera [2], Peng and Yao [3,4], Takahashi and Takahashi [5] considered the following generalized equilibrium problem:

$$\text{Find } x \in C \quad \text{such that} \quad F(x, y) + \langle Bx, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by GEP(F, B).

If $B = 0$, then the problem (1.1) becomes the following equilibrium problem:

$$\text{Find } x \in C \quad \text{such that} \quad F(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The first author was supported by the Grant No. 10831009, Grant No. 10771228, Grant No. 206123, and the Grant No. KJ070816. The second author was partially supported by the grant NSC96-2628-E-110-014-MY3.

J.-W. Peng (✉)
College of Mathematics and Computer Science, Chongqing Normal University,
400047 Chongqing, People's Republic of China
e-mail: jwpeng6@yahoo.com.cn

J.-C. Yao
Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 804, Taiwan, ROC

The set of solutions of (1.2) is denoted by $\mathrm{EP}(F)$.

If $F(x, y) = 0$ for all $x, y \in C$, then the generalized equilibrium problem (1.1) becomes the following variational inequality:

$$\text{Find } x \in C \quad \text{such that} \quad \langle Bx, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $\mathrm{VI}(C, B)$.

The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games and others; see for instance [1–7].

Recall that a mapping $T : C \to C$ is said to be a κ-strict pseudo-contraction mapping [8] if there exists $0 \leq \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C,$$

where I denotes the identity operator on C. When $\kappa = 0$, T is said to be nonexpansive [9], and it is said to be a pseudo-contraction mapping if $\kappa = 1$. It is easy to see that T is a pseudo-contraction mapping if and only if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2, \quad \forall x, y \in C.$$

We denote the set of fixed points of T by $\mathrm{Fix}(T)$.

Some methods have been proposed to solve the problem (1.2); see, for instance [6,7,10–17] and the references therein. Recently, Combettes and Hirstoaga [10] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(F)$ is nonempty and proved a strong convergence theorem. Takahashi and Takahashi [11] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solution of problem (1.2) and the set of fixed points of a nonexpansive mapping and proved a strong convergence theorem in a Hilbert space. Peng and Yao [12] introduced a hybrid iterative scheme for finding the common element of the set of fixed points of a family of infinitely nonexpansive mappings, the set of an equilibrium problem and the set of solutions of variational inequality problem for an α-inverse strongly monotone mapping. Tada and Takahashi [13] introduced some iterative schemes for finding a common element of the set of solution of problem (1.2) and the set of fixed points of a nonexpansive mapping in a Hilbert space and obtained both strong convergence theorem and weak convergence theorem. Ceng et al. [15] introduced an iterative algorithm for finding a common element of the set of solution of problem (1.2) and the set of fixed points of a strict pseudo-contraction mapping. Plubtieng and Punpaeng [16] introduced an iterative process based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings, the set of an equilibrium problem and the set of solutions of variational inequality problem for α-inverse strongly monotone mappings. Chang et al. [17] introduced some iterative processes based on the extragradient method for finding the common element of the set of fixed points of a family of infinitely nonexpansive mappings, the set of an equilibrium problem and the set of solutions of variational inequality problem for an α-inverse strongly monotone mapping.

Several algorithms have also been proposed for finding the solution of problem (1.1). Moudafi [1] introduced an iterative scheme for finding a common element of the set of solutions of problem (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space and proved a weak convergence theorem. Moudafi and Thera [2] introduced an auxiliary scheme for finding a solution of problem (1.1) and obtained a weak convergence theorem. Peng and Yao [3,4] introduced some iterative schemes for finding a common element of the set of solutions of problem (1.1), the set of fixed points of a nonexpansive mapping and the set of the variational inequality for a monotone, Lipschitz-continuous mapping and obtain both strong convergence theorems and weak convergence theorems for the sequences generated by the corresponding processes in Hilbert spaces. Takahashi and Takahashi [5] introduced an iterative scheme for finding a common element of the set of solutions of problem (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space and proved a strong convergence theorem.

On the other hand, Marino and Xu [18] and Zhou [19] introduced and researched some iterative schemes for finding a fixed point of a strict pseudo-contraction mapping and obtained both strong and weak convergence theorems. Ishikawa [20] introduced an Ishikawa iterative algorithm for finding a fixed point of a Lipschitz pseudo-contraction mapping and obtained a weak convergence theorem. Zhou [19] also introduced an Ishikawa iterative algorithm based on hybrid method for finding a fixed point of a Lipschitz pseudo-contraction mapping and obtained a strong convergence theorem.

It is easy to see that a strict pseudo-contraction mapping must be a pseudo-contraction mapping. However, the examples 3 and 4 in [19] illustrate that a Lipschitz and pseudo-contraction mapping may be neither a strict pseudo-contraction mapping nor a nonexpansive mapping. It is natural to raise and to give an answer to the following question: can one construct algorithms for finding a common element of the set of solutions of a generalized equilibrium problem (an equilibrium problem), the common set of fixed points of a Lipschitz pseudo-contraction mapping? In this paper, we will give some positive answers to this question. We introduce some Ishikawa iterative algorithms for finding a common element of the set of solutions of problem (1.1) and the set of fixed points of a Lipschitz continuous pseudo-contraction mapping. We obtain both strong convergence theorems and weak convergence theorems. The results in this paper extend and improve some well-known results in [1,3–5,10–17].

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let symbols $\rightarrow$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping from H onto C. It is known that $P_C(x) \in C$ and

$$\langle x - P_C(x), P_C(x) - y\rangle \geq 0 \tag{2.1}$$

for all $x \in H$ and $y \in C$.

It is easy to see that (2.1) is equivalent to

$$\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2 \tag{2.2}$$

for all $x \in H$ and $y \in C$. It is also known that

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{2.3}$$

Recall that a mapping $A : C \to H$ is called k-Lipschitz continuous if there exists a positive real number k such that

$$\|Ax - Ay\| \leq k\|x - y\|$$

for all $x, y \in C$.

Recall also that a mapping $A : H \to H$ is called $\alpha-$inverse strongly monotone, if there exists an $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in H.$$

For solving the problems (1.1) and (1.2), let us assume that the bifunction F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

3 Strong convergence theorems

In this section, we show a strong convergence of an Ishikawa iterative algorithm based on hybrid method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a Lipschitz continuous pseudo-contraction mapping in a Hilbert space.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A4). *Let B be a β-inverse-strongly monotone mapping of C into H. Let S be L-Lipschitz continuous and a pseudo contraction of C into itself such that $\Omega = \mathrm{Fix}(S) \cap \mathrm{GEP}(F, B) \neq \emptyset$. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = (1-\alpha_n)u_n + \alpha_n Su_n, \\ z_n = (1-\beta_n)u_n + \beta_n Sy_n, \\ C_n = \left\{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n\beta_n\left(1 - 2\alpha_n - L^2\alpha_n^2\right)\|u_n - Su_n\|^2\right\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases}$$

for every $n = 1, 2, \ldots$ where $\{r_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are three sequences satisfying the conditions:

(C1) $\beta_n \leq \alpha_n$ *for all $n \in N$;*
(C2) $\liminf_{n\to\infty} \beta_n > 0$*;*
(C3) $\limsup_{n\to\infty} \alpha_n \leq \alpha < \frac{1}{\sqrt{1+L^2}+1}$ *for all $n \in N$.*
(C4) $\{r_n\} \subset [\gamma, \tau]$ *for some $\gamma, \tau \in (0, 2\beta)$.*

Then, $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $w = P_{\Omega(x)}$.

Proof It is obvious that C_n is closed and Q_n is closed and convex for every $n = 1, 2, \ldots$. Since,

$$C_n = \left\{z \in H : \|z_n - x_n\|^2 + 2\langle z_n - x_n, x_n - z\rangle \right.$$
$$\left. + \alpha_n\beta_n \left(1 - 2\alpha_n - L^2\alpha_n^2\right)\|u_n - Su_n\|^2 \le 0\right\},$$

It follows Lemma 1.3 in [21] that C_n is convex for every $n = 1, 2, \ldots$. It is easy to see that $\langle x_n - z, x - x_n\rangle \ge 0$ for all $z \in Q_n$ and by (2.1), $x_n = P_{Q_n}x$. Let $u \in \Omega$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.2 in [11]. Then $u = T_{r_n}(u - r_nBu)$. From $u_n = T_{r_n}(x_n - r_nBx_n) \in C$, the β-inverse-strongly monotonicity of B and (C4), we have

$$\begin{aligned}\|u_n - u\|^2 &\le \|x_n - r_nBx_n - (u - r_nBu)\|^2\\ &\le \|x_n - u\|^2 - 2r_n\langle x_n - u, Bx_n - Bu\rangle + r_n^2\|Bx_n - Bu\|^2\\ &\le \|x_n - u\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bu\|^2\\ &\le \|x_n - u\|^2. \end{aligned} \tag{3.1}$$

Since $z_n = (1 - \beta_n)u_n + \beta_nSy_n$ and $u = Su$, we have

$$\begin{aligned}\|z_n - u\|^2 &= (1 - \beta_n)\|u_n - u\|^2 + \beta_n\|Sy_n - u\|^2 - (1 - \beta_n)\beta_n\|u_n - Sy_n\|^2\\ &\le (1 - \beta_n)\|u_n - u\|^2 + \beta_n[\|y_n - u\|^2 + \|y_n - Sy_n\|^2]\\ &\quad - (1 - \beta_n)\beta_n\|u_n - Sy_n\|^2. \end{aligned} \tag{3.2}$$

Since S is L-Lipschitz continuous, $y_n = (1 - \alpha_n)u_n + \alpha_nSu_n$ and $u = Su$, we get

$$\begin{aligned}\|y_n - Sy_n\|^2 &= \|(1 - \alpha_n)(u_n - Sy_n) + \alpha_n(Su_n - Sy_n)\|^2\\ &= (1 - \alpha_n)\|u_n - Sy_n\|^2 + \alpha_n\|Su_n - Sy_n\|^2 - (1 - \alpha_n)\alpha_n\|u_n - Su_n\|^2\\ &\le (1 - \alpha_n)\|u_n - Sy_n\|^2 + \alpha_nL^2\|u_n - y_n\|^2 - (1 - \alpha_n)\alpha_n\|u_n - Su_n\|^2\\ &= (1 - \alpha_n)\|u_n - Sy_n\|^2 + \alpha_n^3L^2\|u_n - Su_n\|^2 - (1 - \alpha_n)\alpha_n\|u_n - Su_n\|^2\\ &= (1 - \alpha_n)\|u_n - Sy_n\|^2 + \alpha_n\left(\alpha_n^2L^2 + \alpha_n - 1\right)\|u_n - Su_n\|^2. \end{aligned} \tag{3.3}$$

Since S is a pseudo-contract, we have

$$\begin{aligned}\|y_n - u\|^2 &= (1 - \alpha_n)\|u_n - u\|^2 + \alpha_n\|Su_n - u\|^2 - (1 - \alpha_n)\alpha_n\|u_n - Su_n\|^2\\ &\le (1 - \alpha_n)\|u_n - u\|^2 + \alpha_n\|u_n - u\|^2 + \alpha_n\|u_n - Su_n\|^2\\ &\quad -(1 - \alpha_n)\alpha_n\|u_n - Su_n\|^2\\ &= \|u_n - u\|^2 + \alpha_n^2\|u_n - Su_n\|^2. \end{aligned} \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2) yields

$$\begin{aligned}\|z_n - u\|^2 &\le (1 - \beta_n)\|u_n - u\|^2 + \beta_n\|u_n - u\|^2\\ &\quad + \beta_n\alpha_n^2\|u_n - Su_n\|^2 + \beta_n(1 - \alpha_n)\|u_n - Sy_n\|^2\\ &\quad + \beta_n\alpha_n\left(\alpha_n^2L^2 + \alpha_n - 1\right)\|u_n - Su_n\|^2 - (1 - \beta_n)\beta_n\|u_n - Sy_n\|^2\\ &\le \|u_n - u\|^2 + \beta_n(\beta_n - \alpha_n)\|u_n - Sy_n\|^2\\ &\quad + \beta_n\alpha_n\left(\alpha_n^2L^2 + 2\alpha_n - 1\right)\|u_n - Su_n\|^2. \end{aligned} \tag{3.5}$$

It follows from condition (3.5), (C1), (C3) and (3.1) that

$$\|z_n - u\|^2 \le \|x_n - u\|^2 - \beta_n\alpha_n\left(1 - 2\alpha_n - \alpha_n^2L^2\right)\|u_n - Su_n\|^2 \le \|x_n - u\|^2, \tag{3.6}$$

for every $n = 1, 2, \ldots$ and hence $u \in C_n$. So, $\Omega \subset C_n$ for every $n = 1, 2, \ldots$. Next, let us show by mathematical induction that $\{x_n\}$ is well defined and $\Omega \subset C_n \cap Q_n$ for every

$n = 1, 2, \ldots$. For $n = 1$ we have $x_1 = x \in C$ and $Q_1 = H$. Hence, we obtain $\Omega \subset C_1 \cap Q_1$. Suppose that x_k is given and $\Omega \subset C_k \cap Q_k$ for some $k \in N$. Since Ω is nonempty, $C_k \cap Q_k$ is a nonempty closed convex subset of H. So, there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k} x$. It is also obvious that there holds $\langle x_{k+1} - z, x - x_{k+1} \rangle \geq 0$ for every $z \in C_k \cap Q_k$. Since $\Omega \subset C_k \cap Q_k$, we have $\langle x_{k+1} - z, x - x_{k+1} \rangle \geq 0$ for every $z \in \Omega$ and hence $\Omega \subset Q_{k+1}$. Therefore, we obtain $\Omega \subset C_{k+1} \cap Q_{k+1}$.

Let $l_0 = P_\Omega x$. From $x_{n+1} = P_{C_n \cap Q_n} x$ and $l_0 \in \Omega \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x\| \leq \|l_0 - x\|. \tag{3.7}$$

for every $n = 1, 2, \ldots$. Therefore, $\{x_n\}$ is bounded. From (3.1), (3.4), (3.6) and the Lipschitz continuity of S, we also obtain that $\{u_n\}$, $\{y_n\}$, and $\{z_n\}$ are bounded. Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = P_{Q_n} x$, we have

$$\|x_n - x\| \leq \|x_{n+1} - x\|$$

for every $n = 1, 2, \ldots$. It follows from (3.7) that $\lim_{n\to\infty} \|x_n - x\|$ exists.

Since $x_n = P_{Q_n} x$ and $x_{n+1} \in Q_n$, using (2.2), we have

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x\|^2 - \|x_n - x\|^2$$

for every $n = 1, 2, \ldots$. This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Since $x_{n+1} \in C_n$, we have

$$\begin{aligned} \|z_n - x_{n+1}\|^2 &\leq \|x_n - x_{n+1}\|^2 \\ &\quad - \beta_n \alpha_n \left(1 - 2\alpha_n - \alpha_n^2 L^2\right) \|u_n - Su_n\|^2 \leq \|x_n - x_{n+1}\|^2. \end{aligned} \tag{3.8}$$

Thus, we get $\lim_{n\to\infty} \|z_n - x_{n+1}\| = 0$. It follows from $\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\|$ that $\|x_n - z_n\| \to 0$.

By (3.8), we get

$$\beta_n \alpha_n \left(1 - 2\alpha_n - \alpha_n^2 L^2\right) \|u_n - Su_n\|^2 \leq (\|x_n - x_{n+1}\| + \|z_n - x_{n+1}\|)(\|x_n - z_n\|).$$

It follows from (C1)–(C3), $\{x_n\}$ and $\{z_n\}$ are bounded and $\|x_n - z_n\| \to 0$ that

$$\lim_{n\to\infty} \|u_n - Su_n\| = 0. \tag{3.9}$$

And thus,

$$\lim_{n\to\infty} \|y_n - u_n\| = \lim_{n\to\infty} \alpha_n \|u_n - Su_n\| = 0.$$

By (3.5) and (3.1), we obtain

$$\|z_n - u\|^2 \leq \|u_n - u\|^2 \leq \|x_n - u\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bu\|^2.$$

Thus, we get

$$\begin{aligned} \gamma(2\beta - \tau)\|Bx_n - Bu\|^2 &\leq r_n(2\beta - r_n)\|Bx_n - Bu\|^2 \\ &\leq \|x_n - u\|^2 - \|z_n - u\|^2 \leq (\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|. \end{aligned}$$

It follows from $\|x_n - z_n\| \to 0$, $\{x_n\}$ and $\{z_n\}$ are bounded that $\|Bx_n - Bu\| \to 0$.

For $u \in \Omega$, we have, from Lemma 2.2 in [11],

$$\begin{aligned}\|u_n - u\|^2 &= \|T_{r_n}(x_n - r_n Bx_n) - T_{r_n}(u - r_n Bu)\|^2 \\ &\leq \langle T_{r_n}(x_n - r_n Bx_n) - T_{r_n}(u - r_n Bu), x_n - r_n Bx_n - (u - r_n Bu)\rangle \\ &= \frac{1}{2}\left\{\|u_n - u\|^2 + \|x_n - r_n Bx_n - (u - r_n Bu)\|^2\right. \\ &\quad \left. - \|x_n - r_n Bx_n - (u - r_n Bu) - (u_n - u)\|^2\right\} \\ &\leq \frac{1}{2}\left\{\|u_n - u\|^2 + \|x_n - u\|^2 - \|x_n - r_n Bx_n - (u - r_n Bu) - (u_n - u)\|^2\right\} \\ &= \frac{1}{2}\left\{\|u_n - u\|^2 + \|x_n - u\|^2 - \|x_n - u_n\|^2\right. \\ &\quad \left. + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle - r_n^2\|Bx_n - Bu\|^2\right\}.\end{aligned}$$

Hence,

$$\|u_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle. \tag{3.10}$$

It follows from (3.5) and (3.10) that

$$\|z_n - u\|^2 \leq \|u_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle.$$

Hence,

$$\begin{aligned}\|x_n - u_n\|^2 &\leq \|x_n - u\|^2 - \|z_n - u\|^2 + 2r_n\|Bx_n - Bu\|\|x_n - u_n\|. \\ &\leq (\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\| + 2r_n\|Bx_n - Bu\|\|x_n - u_n\|.\end{aligned}$$

Since $\|Bx_n - Bu\| \to 0$, $\|x_n - z_n\| \to 0$, $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ are bounded, we obtain $\|x_n - u_n\| \to 0$.

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{ni} \rightharpoonup w$. From $\|x_n - u_n\| \to 0$, we obtain that $u_{ni} \rightharpoonup w$. From $\|u_n - y_n\| \to 0$, we also obtain that $y_{ni} \rightharpoonup w$. From $\|x_n - z_n\| \to 0$, we also obtain that $z_{ni} \rightharpoonup w$. Since $\{u_{ni}\} \subset C$ and C is closed and convex, we obtain $w \in C$. It follows from (3.9) that

$$\|u_{n_i} - Su_{n_i}\| \to 0.$$

It follows from Tool 2 in [19] that $w \in \mathrm{Fix}(S)$. We next show that $w \in \mathrm{GEP}(F, B)$. By $u_n = T_{r_n}(x_n - r_n Bx_n)$, we know that

$$F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

It follows from (A2) that

$$\langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n), \quad \forall y \in C.$$

Hence,

$$\langle Bx_{n_i}, y - u_{n_i}\rangle + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq F(y, u_{n_i}), \quad \forall y \in C. \tag{3.11}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we obtain $y_t \in C$. So, from (3.11) we have

$$\begin{aligned}\langle y_t - u_{n_i}, By_t\rangle \geq \langle y_t - u_{n_i}, By_t\rangle - \langle y_t - u_{n_i}, Bx_{n_i}\rangle - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \\ + F(y_t, u_{n_i}) = \langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle + \langle y_t - u_{n_i}, Bu_{n_i} - Bx_{n_i}\rangle \\ - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F(y_t, u_{n_i}).\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Bu_{n_i} - Bx_{n_i}\| \to 0$. Further, from the inverse-strongly monotonicity of B, we have $\langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle \geq 0$. It follows from (A4), $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$ that

$$\langle y_t - w, By_t\rangle \geq F(y_t, w), \tag{3.12}$$

as $i \to \infty$. From (A1), (A4) and (3.12), we also have

$$\begin{aligned}0 &= F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, w) \\ &\leq tF(y_t, y) + (1-t)\langle y_t - w, By_t\rangle = tF(y_t, y) + (1-t)t\langle y - w, By_t\rangle\end{aligned}$$

and hence

$$0 \leq F(y_t, y) + (1-t)\langle y - w, By_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F(w, y) + \langle y - w, Bw\rangle \geq 0.$$

This implies that $w \in \mathrm{GEP}(F, B)$. Hence, we get $w \in \Omega$.

From $l_0 = P_\Omega x$, $w \in \Omega$ and (3.7), we have

$$\|l_0 - x\| \leq \|w - x\| \leq \liminf_{i\to\infty} \|x_{n_i} - x\| \leq \limsup_{i\to\infty} \|x_{n_i} - x\| \leq \|l_0 - x\|.$$

So, we obtain

$$\lim_{i\to\infty} \|x_{n_i} - x\| = \|w - x\|.$$

From $x_{n_i} - x \rightharpoonup w - x$ we have $x_{n_i} - x \to w - x$ and hence $x_{n_i} \to w$. Since $x_n = P_{Q_n}x$ and $l_0 \in \Omega \subset C_n \cap Q_n \subset Q_n$, we have

$$-\|l_0 - x_{n_i}\|^2 = \langle l_0 - x_{n_i}, x_{n_i} - x\rangle + \langle l_0 - x_{n_i}, x - l_0\rangle \geq \langle l_0 - x_{n_i}, x - l_0\rangle.$$

As $i \to \infty$, we obtain $-\|l_0 - w\|^2 \geq \langle l_0 - w, x - l_0\rangle \geq 0$ by $l_0 = P_\Omega x$ and $w \in \Omega$. Hence we have $w = l_0$. This implies that $x_n \to l_0$. It is easy to see $u_n \to l_0$, $y_n \to l_0$ and $z_n \to l_0$. The proof is now complete. □

By Theorem 3.1, we can obtain the following new and interesting strong convergence theorems in a real Hilbert space.

Corollary 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A4). *Let S be L-Lipschitz continuous and* *a pseudo contraction of C into itself such that* $\mathrm{Fix}(S) \cap \mathrm{EP}(F) \neq \emptyset$. *Let* $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = (1-\alpha_n)u_n + \alpha_n S u_n, \\ z_n = (1-\beta_n)u_n + \beta_n S y_n, \\ C_n = \left\{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n\beta_n\left(1 - 2\alpha_n - L^2\alpha_n^2\right)\|u_n - Su_n\|^2\right\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases}$$

for every $n = 1, 2, \ldots$ *where* $\{r_n\}$, $\{\alpha_n\}$, *and* $\{\beta_n\}$ *are three sequences satisfying the conditions*:

(C1) $\beta_n \leq \alpha_n$ *for all* $n \in N$;
(C2) $\liminf_{n\to\infty} \beta_n > 0$;
(C3) $\lim\sup_{n\to\infty} \alpha_n \leq \alpha < \frac{1}{\sqrt{1+L^2}+1}$ *for all* $n \in N$.
(C4) $\{r_n\} \subset [\gamma, +\infty)$ *for some* $\gamma > 0$.

Then, $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to* $w = P_{\mathrm{Fix}(S)\cap \mathrm{EP}(F)}(x)$.

Proof Putting $B = 0$, by Theorem 3.1 we obtain the desired result. □

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let B be a β-inverse-strongly monotone mapping of C into H. Let S be L-Lipschitz continuous and a pseudo contraction of C into itself such that* $\mathrm{Fix}(S) \cap VI(B) \neq \emptyset$. *Let* $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ u_n = P_C(x_n - r_n B x_n), \\ y_n = (1-\alpha_n)u_n + \alpha_n S u_n, \\ z_n = (1-\beta_n)u_n + \beta_n S y_n, \\ C_n = \left\{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n\beta_n\left(1 - 2\alpha_n - L^2\alpha_n^2\right)\|u_n - Su_n\|^2\right\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases}$$

for every $n = 1, 2, \ldots$ *where* $\{r_n\}$, $\{\alpha_n\}$, *and* $\{\beta_n\}$ *are three sequences satisfying the conditions*:

(C1) $\beta_n \leq \alpha_n$ *for all* $n \in N$;
(C2) $\liminf_{n\to\infty} \beta_n > 0$;
(C3) $\lim\sup_{n\to\infty} \alpha_n \leq \alpha < \frac{1}{\sqrt{1+L^2}+1}$ *for all* $n \in N$.
(C4) $\{r_n\} \subset [\gamma, \tau]$ *for some* $\gamma, \tau \in (0, 2\beta)$.

Then, $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to* $w = P_{\mathrm{Fix}(S)\cap VI(B)}(x)$.

Proof In Theorem 3.1, put $F = 0$. Then, we obtain that

$$\langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \quad \forall n \in N.$$

This implies that

$$\langle y - u_n, u_n - (x_n - r_n B x_n)\rangle \geq 0, \quad \forall y \in C, \quad \forall n \in N.$$

So, we get that $u_n = P_C(x_n - r_n Bx_n)$ for all $n \in N$. Then we obtain the desired result from Theorem 3.1. □

Corollary 3.3 (i.e., Theorem 3.6 in [19]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S be L-Lipschitz continuous and a pseudo contraction of C into itself such that* $\mathrm{Fix}(S) \neq \emptyset$. *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ y_n = (1-\alpha_n)x_n + \alpha_n Sx_n, \\ z_n = (1-\beta_n)x_n + \beta_n Sy_n, \\ C_n = \left\{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n\beta_n\left(1 - 2\alpha_n - L^2\alpha_n^2\right)\|x_n - Sx_n\|^2\right\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases}$$

for every $n = 1, 2, \ldots$ where $\{r_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are three sequences satisfying the conditions:

(C1) $\beta_n \leq \alpha_n$ *for all $n \in N$;*
(C2) $\liminf_{n\to\infty} \beta_n > 0$;
(C3) $\limsup_{n\to\infty} \alpha_n \leq \alpha < \frac{1}{\sqrt{1+L^2}+1}$ *for all $n \in N$.*

Then, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $w = P_{\mathrm{Fix}(S)}(x)$.

Proof Putting $F = 0$ and $B = 0$, then $u_n = P_C x_n = x_n$, by Theorem 3.1 we obtain the desired result. □

Corollary 3.4 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\varphi : C \to R$ be a lower semicontinuous and convex function. Let S be L-Lipschitz continuous and a pseudo contraction of C into itself such that* $\mathrm{Fix}(S) \cap \mathrm{argmin}(\varphi) \neq \emptyset$. *Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ u_n = \mathrm{argmin}_{y\in C}\{\varphi(y) + \frac{1}{2r_n}\|y - x_n\|^2\}, \\ y_n = (1-\alpha_n)u_n + \alpha_n Su_n, \\ z_n = (1-\beta_n)u_n + \beta_n Sy_n, \\ C_n = \left\{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n\beta_n\left(1 - 2\alpha_n - L^2\alpha_n^2\right)\|u_n - Su_n\|^2\right\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases}$$

for every $n = 1, 2, \ldots$ where $\{r_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are three sequences satisfying the conditions:

(C1) $\beta_n \leq \alpha_n$ *for all $n \in N$;*
(C2) $\liminf_{n\to\infty} \beta_n > 0$;
(C3) $\limsup_{n\to\infty} \alpha_n \leq \alpha < \frac{1}{\sqrt{1+L^2}+1}$ *for all $n \in N$.*
(C4) $\{r_n\} \subset [\gamma, +\infty)$ *for some* $\gamma > 0$.

Then, $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $w = P_{\mathrm{Fix}(S)\cap\mathrm{argmin}(\varphi)}(x)$.

Proof Putting $F(x, y) = \varphi(y) - \varphi(x)$, $\forall x, y \in C$. by Theorem 3.1 we obtain the desired result. □

4 Weak convergence theorems

In this section, we show a weak convergence of an Ishikawa iterative algorithm for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a Lipschitz continuous pseudo-contraction mapping in a Hilbert space.

Theorem 4.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A4). *Let B be a β-inverse-strongly monotone mapping of C into H. Let S be L-Lipschitz continuous and a pseudo contraction of C into itself such that $\Omega = \mathrm{Fix}(S) \cap \mathrm{GEP}(F, B) \neq \emptyset$. Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = (1-\alpha_n)u_n + \alpha_n S u_n, \\ x_{n+1} = (1-\beta_n)u_n + \beta_n S y_n, \end{cases} \tag{4.1}$$

for every $n = 1, 2, \ldots$ where $\{r_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are three sequences satisfying the conditions:

(C1) $\beta_n \leq \alpha_n$ *for all $n \in N$*;
(C2) $\liminf_{n\to\infty} \beta_n > 0$;
(C3) $\limsup_{n\to\infty} \alpha_n \leq \alpha < \frac{1}{\sqrt{1+L^2}+1}$ *for all $n \in N$.*
(C4) $\{r_n\} \subset [\gamma, \tau]$ *for some $\gamma, \tau \in (0, 2\beta)$.*

Then, $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge weakly to $w = P_\Omega(x_n)$.

Proof Let $u \in \Omega$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.2 in [11]. From $u_n = T_{r_n}(x_n - r_n Bx_n) \in C$, the algorithm (4.1) and the proof of Theorem 3.1, we have

$$\|u_n - u\|^2 \leq \|x_n - u\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bu\|^2 \leq \|x_n - u\|^2. \tag{3.1}$$

$$\|y_n - Sy_n\|^2 \leq (1-\alpha_n)\|u_n - Sy_n\|^2 + \alpha_n\left(\alpha_n^2 L^2 + \alpha_n - 1\right)\|u_n - Su_n\|^2. \tag{3.3}$$

$$\|y_n - u\|^2 \leq \|u_n - u\|^2 + \alpha_n^2\|u_n - Su_n\|^2. \tag{3.4}$$

And

$$\|u_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle. \tag{3.10}$$

Since $x_{n+1} = (1-\beta_n)u_n + \beta_n S y_n$ and $u = Su$, we have

$$\begin{aligned} \|x_{n+1} - u\|^2 \leq\ & (1-\beta_n)\|u_n - u\|^2 + \beta_n\left[\|y_n - u\|^2 + \|y_n - Sy_n\|^2\right] \\ & -(1-\beta_n)\beta_n\|u_n - Sy_n\|^2. \end{aligned} \tag{4.2}$$

Substituting (3.3) and (3.4) into (4.2) yields

$$\begin{aligned} \|x_{n+1} - u\|^2 \leq\ & (1-\beta_n)\|u_n - u\|^2 + \beta_n\|u_n - u\|^2 + \beta_n\alpha_n^2\|u_n - Su_n\|^2 \\ & + \beta_n(1-\alpha_n)\|u_n - Sy_n\|^2 \\ & + \beta_n\alpha_n\left(\alpha_n^2L^2 + \alpha_n - 1\right)\|u_n - Su_n\|^2 - (1-\beta_n)\beta_n\|u_n - Sy_n\|^2 \\ \leq\ & \|u_n - u\|^2 + \beta_n(\beta_n - \alpha_n)\|u_n - Sy_n\|^2 \\ & + \beta_n\alpha_n\left(\alpha_n^2L^2 + 2\alpha_n - 1\right)\|u_n - Su_n\|^2. \end{aligned} \tag{4.3}$$

It follows from (4.3), (C1)–(C3) and (3.1) that

$$\|x_{n+1}-u\|^2 \le \|x_n-u\|^2 - \beta_n\alpha_n\left(1-2\alpha_n-\alpha_n^2L^2\right)\|u_n-Su_n\|^2 \le \|x_n-u\|^2, \quad (4.4)$$

for every $n=1,2,\ldots$. Therefore, there exists $\theta=\lim_{n\to\infty}\|x_n-u\|$ and $\{x_n\}$ is bounded. From (3.1) and (3.4), we also know that $\{u_n\}$ and $\{y_n\}$ are bounded.

By (4.3), (C1)–(C3) and (3.1), we have

$$\|x_{n+1}-u\|^2 \le \|u_n-u\|^2 \le \|x_n-u\|^2 + r_n(r_n-2\beta)\|Bx_n-Bu\|^2.$$

It follows that

$$\gamma(2\beta-\tau)\|Bx_n-Bu\|^2 \le r_n(2\beta-r_n)\|Bx_n-Bu\|^2 \le \|x_n-u\|^2-\|x_{n+1}-u\|^2.$$

So, we obtain $\|Bx_n-Bu\|\to 0$.

By (4.3), (C1)–(C3) and (3.10), we have

$$\|x_{n+1}-u\|^2 \le \|u_n-u\|^2 \le \|x_n-u\|^2-\|x_n-u_n\|^2+2r_n\langle Bx_n-Bu, x_n-u_n\rangle.$$

Hence,

$$\|x_n-u_n\|^2 \le \|x_n-u\|^2-\|x_{n+1}-u\|^2+2r_n\|Bx_n-Bu\|\|x_n-u_n\|.$$

Since $\|Bx_n-Bu\|\to 0$, $\{x_n\}$ and $\{u_n\}$ are bounded, we obtain $\|x_n-u_n\|\to 0$.

By (4.4), we get

$$\beta_n\alpha_n(1-2\alpha_n-\alpha_n^2L^2)\|u_n-Su_n\|^2 \le \|x_n-u\|^2-\|x_{n+1}-u\|^2.$$

It follows from (C1)–(C3) that $\|u_n-Su_n\|\to 0$.

And thus,

$$\lim_{n\to\infty}\|y_n-u_n\| = \alpha_n\|u_n-Su_n\| = 0.$$

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{ni}\rightharpoonup w$. From $\|x_n-u_n\|\to 0$, we obtain that $u_{ni}\rightharpoonup w$. From $\|u_n-y_n\|\to 0$, we also obtain that $y_{ni}\rightharpoonup w$. Since $\{u_{ni}\}\subset C$ and C is closed and convex, we obtain $w\in C$. It follows from $\|u_{n_i}-Su_{n_i}\|\to 0$ and Tool 2 in [19] that $w\in \mathrm{Fix}(S)$. By similar argument with that in the proof of Theorem 3.1, we can easily show that $w\in \mathrm{GEP}(F,B)$. This implies $w\in\Omega$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j}\rightharpoonup z$. Then $z\in\Omega$. Let us show $w=z$. Assume that $w\ne z$. From the Opial condition, we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n-w\| &= \liminf_{i\to\infty}\|x_{n_i}-w\| < \liminf_{i\to\infty}\|x_{n_i}-z\| \\ &= \lim_{n\to\infty}\|x_n-z\| = \liminf_{j\to\infty}\|x_{n_j}-z\| \\ &< \liminf_{j\to\infty}\|x_{n_j}-w\| = \lim_{n\to\infty}\|x_n-w\|.\end{aligned}$$

This is a contradiction. Thus, we have $w=z$. This implies that $x_n\rightharpoonup w\in\Omega$. Since $\|x_n-u_n\|\to 0$, we have $u_n\rightharpoonup w\in\Omega$. Since $\|y_n-u_n\|\to 0$, we have also $y_n\rightharpoonup w\in\Omega$.

Now put $w_n=P_\Omega x_n$. We show that $w=\lim_{n\to\infty}w_n$.

From $w_n=P_\Omega x_n$ and $w\in\Omega$, we have

$$\langle w-w_n, w_n-x_n\rangle \ge 0.$$

From (4.4) and Lemma 4.2 in [13], we know that $\{w_n\}$ converges strongly to some $w_0\in\Omega$. Then, we have

$$\langle w - w_0, w_0 - w\rangle \geq 0$$

and hence $w = w_0$. The proof is now complete. □

By Theorem 4.1 and similar arguments with those in the proof of Corollaries 3.1–3.4, we can obtain the following new and interesting weak convergence theorems in a real Hilbert space.

Corollary 4.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A4). *Let S be L-Lipschitz continuous and a pseudo contraction of C into itself such that* $\mathrm{Fix}(S) \cap \mathrm{EP}(F) \neq \emptyset$. *Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = (1-\alpha_n)u_n + \alpha_n S u_n, \\ x_{n+1} = (1-\beta_n)u_n + \beta_n S y_n, \end{cases}$$

for every $n = 1, 2, \ldots$ where $\{r_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are three sequences satisfying the conditions:

(C1) $\beta_n \leq \alpha_n$ *for all $n \in N$;*
(C2) $\liminf_{n\to\infty} \beta_n > 0$*;*
(C3) $\lim\sup_{n\to\infty} \alpha_n \leq \alpha < \frac{1}{\sqrt{1+L^2}+1}$ *for all $n \in N$.*
(C4) $\{r_n\} \subset [\gamma, +\infty)$ *for some $\gamma > 0$.*

Then, $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge weakly to $w = P_{\mathrm{Fix}(S)\cap\mathrm{EP}(F)}(x_n)$.

Corollary 4.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A4). *Let B be a β-inverse-strongly monotone mapping of C into H. Let S be L-Lipschitz continuous and a pseudo contraction of C into itself such that* $\mathrm{Fix}(S) \cap VI(B) \neq \emptyset$. *Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ u_n = P_C(x_n - r_n B x_n), \\ y_n = (1-\alpha_n)u_n + \alpha_n S u_n, \\ x_{n+1} = (1-\beta_n)u_n + \beta_n S y_n, \end{cases}$$

for every $n = 1, 2, \ldots$ where $\{r_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are three sequences satisfying the conditions:

(C1) $\beta_n \leq \alpha_n$ *for all $n \in N$*;
(C2) $\liminf_{n\to\infty} \beta_n > 0$;
(C3) $\lim\sup_{n\to\infty} \alpha_n \leq \alpha < \frac{1}{\sqrt{1+L^2}+1}$ *for all $n \in N$.*
(C4) $\{r_n\} \subset [\gamma, \tau]$ *for some $\gamma, \tau \in (0, 2\beta)$.*

Then, $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ converge weakly to $w = P_{\mathrm{Fix}(S)\cap VI(B)}(x_n)$.

Corollary 4.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S be L-Lipschitz continuous and a pseudo contraction of C into itself such that* $\mathrm{Fix}(S) \neq \emptyset$. *Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ y_n = (1-\alpha_n)x_n + \alpha_n S x_n, \\ x_{n+1} = (1-\beta_n)x_n + \beta_n S y_n, \end{cases} \tag{4.1}$$

for every $n = 1, 2, \dots$ *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are two sequences satisfying the conditions*:

(C1) $\beta_n \leq \alpha_n$ *for all* $n \in N$;
(C2) $\liminf_{n\to\infty} \beta_n > 0$;
(C3) $\lim\sup_{n\to\infty} \alpha_n \leq \alpha < \frac{1}{\sqrt{1+L^2}+1}$ *for all* $n \in N$.

Then, $\{x_n\}$ *and* $\{y_n\}$ *converge weakly to* $w = P_{\mathrm{Fix}(S)}(x_n)$.

Corollary 4.4 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\varphi : C \to R$ *be a lower semicontinuous and convex function. Let* S *be* L*-Lipschitz continuous and a pseudo contraction of* C *into itself such that* $\mathrm{Fix}(S) \cap \mathrm{argmin}(\varphi) \neq \emptyset$. *Let* $\{x_n\}$, $\{u_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ u_n = \mathrm{argmin}_{y\in C}\left\{\varphi(y) + \frac{1}{2r_n}\|y - x_n\|^2\right\}, \\ y_n = (1-\alpha_n)u_n + \alpha_n S u_n, \\ x_{n+1} = (1-\beta_n)u_n + \beta_n S y_n, \end{cases} \tag{4.1}$$

for every $n = 1, 2, \dots$ *where* $\{r_n\}$, $\{\alpha_n\}$, *and* $\{\beta_n\}$ *are three sequences satisfying the conditions*:

(C1) $\beta_n \leq \alpha_n$ *for all* $n \in N$;
(C2) $\liminf_{n\to\infty} \beta_n > 0$;
(C3) $\lim\sup_{n\to\infty} \alpha_n \leq \alpha < \frac{1}{\sqrt{1+L^2}+1}$ *for all* $n \in N$.
(C4) $\{r_n\} \subset [\gamma, \tau]$ *for some* $\gamma, \tau \in (0, 2\beta)$.

Then, $\{x_n\}$, $\{u_n\}$ *and* $\{y_n\}$ *converge weakly to* $w = P_\Omega(x_n)$.

Remark 4.1

(i) Since the nonexpansive mappings or strict pseudo-contraction mappings has been replaced by a pseudo-contraction mapping, Theorem 3.1 improves Theorems 4.3 and 4.4 in [3], and Theorem 3.1 in [5]. Theorem 4.1 improves Theorem 3.1 in [1] and [4]. It is easy to see that Theorems 3.1 and 4.1, Corollaries 3.1 and 4.1 extend and improve the corresponding results in [10–17].
(ii) The algorithm in Corollaries 3.4 and 4.4 are variants of the proximal method for optimization problems introduced and studied by Martinet [22], Rockafellar [23], Ferris [24] and many others.

Acknowledgments The authors would like to express their thanks to the referees for helpful suggestions.

References

1. Moudafi, A.: Weak convergence theorems for nonexpansive mappings and equilibrium problems. J. Nonlinear Convex Anal. **9**, 37–43 (2008)
2. Moudafi, A., Thera, M.: Proximal and dynamical approaches to equilibrium problems. Lecture Notes in Econom. and Math. System, vol. 477, pp. 187–201. Springer, Berlin (1999)
3. Peng, J.W., Yao, J.C.: A new hybrid-extragradient method for generalized mixed equilibrium problems and fixed point Problems and variational inequality problems. Taiwan. J. Math. **12**(6), 1401–1432 (2008)
4. Peng, J.W., Yao, J.C.: Weak convergence of an iterative scheme for generalized equilibrium problems. Bull. Aust. Math. Soc. (preprint)

5. Takahashi, S., Takahashi, W.: Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space. Nonlinear Anal. **69**, 1025–1033 (2008)
6. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)
7. Flam, S.D., Antipin, A.S.: Equilibrium programming using proximal-like algorithms. Math. Program. **78**, 29–41 (1997)
8. Browder, F.E., Petryshyn, W.V.: Construction of fixed points of nonlinear mappings in Hilbert spaces. J. Math. Anal. Appl. **20**, 197–228 (1967)
9. Goebel, K., Kirk, W.A.: Topics on Metric Fixed-Point Theory. Cambridge University Press, Cambridge (1990)
10. Combettes, P.L., Hirstoaga, S.A.: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6**, 117–136 (2005)
11. Takahashi, S., Takahashi, W.: Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces. J. Math. Anal. Appl. **331**, 506–515 (2007)
12. Peng, J.W., Yao, J.C.: A modified CQ method for equilibrium problems and fixed point problems and variational inequality problems. Fixed Point Theory **9**, 515–531 (2008)
13. Tada, A., Takahashi, W.: Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem. J. Optim. Theory Appl. **133**, 359–370 (2007)
14. Colao, V., Marino, G., Xu, H.K.: An iterative method for finding common solutions of equilibrium and fixed point problems. J. Math. Anal. Appl. **344**, 340–352 (2008)
15. Ceng, L.C., AI-Homidan, S., Ansari, Q.H., Yao, J.C.: An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings. J. Comp. Appl. Math. **223**, 967–974 (2009)
16. Plubtieng, S., Punpaeng, R.: A new iterative method for equilibrium problems and fixed point problems of nonexpansive mappings and monotone mappings. Appl. Math. Comput. **197**, 548–558 (2008)
17. Chang, S.S., Joseph Lee, H.W., Chan, C.K.: A new method for solving equilibrium problem, fixed point problem and variational inequality problem with application to optimization. Nonlinear Anal. (2008). doi:10.1016/j.na.2008.04.035
18. Marino, G., Xu, H.K.: Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. J. Math. Anal. Appl. **329**, 336–346 (2007)
19. Zhou, H.Y.: Convergence theorems of fixed points for κ-strict pseudo-contractions in Hilbert spaces. J. Math. Anal. Appl. **343**, 546–556 (2008)
20. Ishikawa, S.: Fixed points by a new iteration method. Proc. Am. Math. Soc. **44**, 147–150 (1974)
21. Martinez-Yanes, C., Xu, H.-K.: Strong convergence of the CQ method for fixed point iteration processes. Nonlinear Anal. **64**, 2400–2411 (2006)
22. Martinet, B.: Perturbation des méthodes d'optimisation. RAIRO Anal. Numer. **12**, 153–171 (1978)
23. Rockafellar, R.T.: Monotone operators and the proximal point algorithm. SIAM J. Control Optim. **14**, 877–898 (1976)
24. Ferris, M.C.: Finite termination of the proximal point algorithm. Math. Program. **50**, 359–366 (1991)